

ADDITIVE AND MULTIPLICATIVE STRUCTURE OF C*-SETS

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ABSTRACT. It is known that for an IP* set A in \mathbb{N} and a sequence $\langle x_n \rangle_{n=1}^\infty$ there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that $FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A$. Similar types of results also have been proved for central* sets where the sequences have been taken from the class of minimal sequences. In this present work we will prove some analogues results for C*-sets for a more general class of sequences.

1. INTRODUCTION

One of the famous Ramsey theoretic results is Hindman's Theorem.

Theorem 1.1. *Given a finite coloring $\mathbb{N} = \bigcup_{i=1}^r A_i$ there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} and $i \in \{1, 2, \dots, r\}$ such that*

$$FS(\langle x_n \rangle_{n=1}^\infty) = \left\{ \sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \right\} \subseteq A_i,$$

where for any set X , $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X .

A strongly negative answer to the partition analogue of Hindman's theorem was presented in [9]. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} , let us denote $PS(\langle x_n \rangle_{n=1}^\infty) = \{x_m + x_n : m, n \in \mathbb{N} \text{ and } m \neq n\}$ and $PP(\langle x_n \rangle_{n=1}^\infty) = \{x_m \cdot x_n : m, n \in \mathbb{N} \text{ and } m \neq n\}$.

Theorem 1.2. *There exists a finite partition \mathcal{R} of \mathbb{N} with no one-to one sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} such that $PS(\langle x_n \rangle_{n=1}^\infty) \cup PP(\langle x_n \rangle_{n=1}^\infty)$ is contained in one cell of the partition \mathcal{R} .*

Proof. [9, Theorem 2.11]. □

The original proof of Theorem 1.1 was combinatorial in nature. But latter using algebraic structure of $\beta\mathbb{N}$ a very elegant proof of this Theorem was established in [11, Corollary 5.10]. First we give a brief description of algebraic structure of βS_d for a discrete semigroup (S, \cdot) .

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We take the points of βS_d to be the ultrafilters on S , identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S_d$. Given $A \subseteq S$,

$$clA = \overline{A} = \{p \in \beta S_d : A \in p\}$$

is a basis for the closed sets of βS_d . The operation \cdot on S can be extended to the Stone-Ćech compactification βS_d of S so that $(\beta S_d, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S_d$, the function $\rho_p : \beta S_d \rightarrow \beta S_d$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S_d \rightarrow \beta S_d$ defined by $\lambda_x(q) = x \cdot q$ is continuous). A nonempty subset I of a semigroup T is called a *left ideal* of S if $TI \subset I$, a *right ideal* if $IT \subset I$, and a *two sided ideal* (or simply an *ideal*) if it is both a left and right ideal. A *minimal left ideal* is the left ideal that does not contain any proper left ideal. Similarly, we can define *minimal right ideal* and *smallest ideal*.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal

$$\begin{aligned} K(T) &= \bigcup \{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup \{R : R \text{ is a minimal right ideal of } T\} \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R , $L \cap R$ is a group, and in particular contains an idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of T .

Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if the set $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [11] for an elementary introduction to the algebra of βS and for any unfamiliar details.

$A \subseteq \mathbb{N}$ is called an IP^* set if it belongs to every idempotent in $\beta \mathbb{N}$. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} , we let $FP(\langle x_n \rangle_{n=1}^\infty)$ be the product analogue of Finite Sum. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} , we say that $\langle y_n \rangle_{n=1}^\infty$ is a *sum subsystem* of $\langle x_n \rangle_{n=1}^\infty$ provided there is a sequence $\langle H_n \rangle_{n=1}^\infty$ of nonempty finite subsets of \mathbb{N} such that $\max H_n < \min H_{n+1}$ and $y_n = \sum_{t \in H_n} x_t$ for each $n \in \mathbb{N}$. The following theorem shows that IP^* -sets have substantial multiplicative structure.

Theorem 1.3. *Let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in \mathbb{N} and A be an IP^* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that*

$$FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A.$$

Proof. [3, Theorem 2.6] or see [11, Corollary 16.21]. \square

Definition 1.4. A subset $C \subseteq S$ is called central if and only if there is an idempotent $p \in K(\beta S)$ such that $C \in p$.

The algebraic structure of the smallest ideal of βS has played a significant role in Ramsey Theory. It is known that any central subset of $(\mathbb{N}, +)$ is guaranteed to have substantial combinatorial additive structure. But Theorem

16.27 of [11] shows that central sets in $(\mathbb{N}, +)$ need not have any multiplicative structure at all. On the other hand, in [3] we see that sets which belong to every minimal idempotent of \mathbb{N} , called central* sets, must have significant multiplicative structure. In fact central* sets in any semigroup (S, \cdot) are defined to be those sets which meet every every central set.

Theorem 1.5. *If A is a central* set in $(\mathbb{N}, +)$ then it is central in (\mathbb{N}, \cdot) .*

Proof. [3, Theorem 2.4]. \square

In case of central* sets a similar result has been proved in [4] for a restricted class of sequences called minimal sequences, where a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} is said to be a minimal sequence if

$$\bigcap_{m=1}^{\infty} \overline{FS(\langle x_n \rangle_{n=m}^\infty)} \cap K(\beta\mathbb{N}) \neq \emptyset.$$

Theorem 1.6. *Let $\langle y_n \rangle_{n=1}^\infty$ be a minimal sequence and A be a central* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle x_n \rangle_{n=1}^\infty$ of $\langle y_n \rangle_{n=1}^\infty$ such that*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A.$$

Proof. [4, Theorem 2.4]. \square

Similar result has been proved in different setup in [6].

The original Central Sets Theorem was proved by Furstenberg in [7, Theorem 8.1] (using a different but equivalent definition of central set). However most general version of Central Sets Theorem is in [5]. We state it only for commutative semigroup.

Theorem 1.7. *Let (S, \cdot) be a commutative semigroup and let $\mathcal{T} = {}^\mathbb{N}S$, the set of sequences in S . Let C be a central subset of S . There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \rightarrow S$ and $H : \mathcal{P}_f(\mathcal{T}) \rightarrow \mathcal{P}_f(\mathbb{N})$ such that*

- (1) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*
- (2) *whenever $m \in \mathbb{N}$, $G_1, G_2, \dots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_m$, and for each $i \in \{1, 2, \dots, m\}$, $f_i \in G_i$, one has*

$$\prod_{i=1}^m (\alpha(G_i) \cdot \prod_{t \in H(G_i)} f_i(t)) \in C$$

Recently those sets have been paid attention lots of attention which satisfies the conclusion of latest Central Sets Theorem.

Definition 1.8. Let (S, \cdot) be a commutative semigroup and let $\mathcal{T} = {}^\mathbb{N}S$, the set of sequences in S . A subset C of S is said to be C -set if there exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \rightarrow S$ and $H : \mathcal{P}_f(\mathcal{T}) \rightarrow \mathcal{P}_f(\mathbb{N})$ such that

- (1) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*

- (2) whenever $m \in \mathbb{N}$, $G_1, G_2, \dots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_m$, and for each $i \in \{1, 2, \dots, m\}$, $f_i \in G_i$, one has

$$\prod_{i=1}^m (\alpha(G_i) \cdot \prod_{t \in H(G_i)} f_i(t)) \in C$$

We now introduce some notation from [5].

Definition 1.9. Let (S, \cdot) be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$ the set of sequences in S .

- (1) A subset A of S is said to be J -set if and only for every $F \in \mathcal{P}_f(\mathcal{T})$ there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$, $a \cdot \prod_{t \in H} f(t) \in A$.
(2) $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a } J\text{-set})\}$.

Theorem 1.10. Let (S, \cdot) be a discrete commutative semigroup and $A \subseteq S$. Then A is a J -set if and only if $J(S) \cap \text{cl}A \neq \emptyset$.

Proof. Since J -sets are partition regular family, hence the theorem follows from [11, Theorem 3.11]. \square

The following is a consequence of [5, Theorem 3.8]. The easy proof for commutative case can be found in [12, Theorem 2.5].

Theorem 1.11. Let $(S, +)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$ the set of sequences in S , and let $A \subseteq S$. Then A is a C -set if and only if there is an idempotent $p \in \text{cl}A \cap J(S)$.

We conclude this introductory discussion with the following [5, Theorem 3.5].

Theorem 1.12. If (S, \cdot) be a discrete commutative semigroup then $J(S)$ is a closed two sided ideal of βS and $\text{cl}K(\beta S) \subset J(S)$.

2. C^* SET

Like IP^* and Central^* sets we introduce the notion of C^* set.

Definition 2.1. Let (S, \cdot) be a discrete commutative semigroup. A set $A \subseteq S$ is said to be C^* set if it contains all the idempotents of $J(S)$

The following implication is clear from the definition of C^* set that

$$\text{IP}^*\text{-set} \Rightarrow C^*\text{-set} \Rightarrow \text{Central}^*\text{-set}$$

Remark 2.2. It is shown in [10] that there exists a set A which is a C -set in $(\mathbb{N}, +)$ with upper Banach density 0. Since A is a C -set in $(\mathbb{N}, +)$ there exists an idempotent $p \in J(\mathbb{N})$ such that $A \in p$. But as upper Banach density of A is 0 it is not central set in $(\mathbb{N}, +)$, so that contains no minimal idempotent of $\beta\mathbb{N}$. Hence $\mathbb{N} \setminus A$ contains all the minimal idempotents in $\beta\mathbb{N}$ that is it is a Central^* -set but not a C^* -set as $\mathbb{N} \setminus A \notin p$.

Definition 2.3. Let $(S, +)$ be a discrete commutative semigroup. A sequence $\langle x_n \rangle_{n=1}^\infty$ is said to be *almost minimal sequence* if

$$\bigcap_{m=1}^\infty \text{clFS}(\langle x_n \rangle_{n=m}^\infty) \cap J(S) \neq \emptyset.$$

To produce an example of an *almost minimal sequence* which is not minimal, let us recall the following notion of “Banach density”. The author is thankful to Prof. Neil Hindman for his help to construct this example.

Definition 2.4. Let $A \subset \mathbb{N}$. Then

- (1) $d^*(A) = \sup\{\alpha \in \mathbb{R} : (\forall k \in \mathbb{N})(\exists n > k)(\exists a \in \mathbb{N})(|A \cap \{a+1, a+2, \dots, a+n\}| \geq \alpha \cdot n)\}$.
- (2) $\Delta^* = \{p \in \beta\mathbb{N} : (\forall A \in p)(d^*(A) > 0)\}$.

Δ^* being a closed ideal we have $\text{cl}K(\beta\mathbb{N}) \subset \Delta^*$. Since every $A \subset \mathbb{N}$ with positive “Banach density” is also a J -set we have $\Delta^* \subset J(\mathbb{N})$. In [10] author has been proved that the above inequality is trivial by producing a C -set with zero “Banach density”. Now to produce an example of an almost minimal sequence which is not minimal let us recall the following theorem.

Theorem 2.5. Let $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} such that for all $n \in \mathbb{N}$, $x_{n+1} > \sum_{t=1}^n x_t$ and $T = \bigcap_{m=1}^\infty \text{clFS}(\langle x_n \rangle_{n=m}^\infty)$. Then the following conditions are equivalent:

- (1) $T \cap K(\beta\mathbb{N}) \neq \emptyset$.
- (2) $T \cap \text{cl}K(\beta\mathbb{N}) \neq \emptyset$.
- (3) $\{x_{n+1} - \sum_{t=1}^n x_t\}$ is bounded.

Proof. [1, Theorem 2.1] □

Now let $d \in \mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ be defined in \mathbb{N} by the formula $x_{n+1} = \sum_{t=1}^n x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$. Then $\{x_{n+1} - \sum_{t=1}^n x_t\}$ being unbounded we have $\bigcap_{m=1}^\infty \text{clFS}(\langle x_n \rangle_{n=m}^\infty) \cap K(\beta\mathbb{N}) = \emptyset$. Therefore the sequence $\langle x_n \rangle_{n=1}^\infty$ is not minimal. But by [1, Lemma 2.20] we have $\bigcap_{m=1}^\infty \text{clFS}(\langle x_n \rangle_{n=m}^\infty) \cap \Delta^* \neq \emptyset$. Since $\Delta^* \subset J(\mathbb{N})$ this implies that this sequence $\langle x_n \rangle_{n=1}^\infty$ is *almost minimal*.

Question 2.6. Does there exist a sequence $\langle x_n \rangle_{n=1}^\infty$ in \mathbb{N} with the property that $\text{FS}(\langle x_n \rangle_{n=1}^\infty)$ is a C -set but its upper banach density is zero.

Theorem 2.7. In the commutative semigroup $(\mathbb{N}, +)$ the following conditions are equivalent:

- (a) $\langle x_n \rangle_{n=1}^\infty$ is an almost minimal sequence.
- (b) $\text{FS}(\langle x_n \rangle_{n=1}^\infty)$ is J -set.
- (c) There is an idempotent in $\bigcap_{m=1}^\infty \text{clFS}(\langle x_n \rangle_{n=m}^\infty) \cap J(\mathbb{N}) \neq \emptyset$.

Proof. (a) \Rightarrow (b) follows from the definition of almost minimal sequence.

(b) \Rightarrow (a) Let $\text{FS}(\langle x_n \rangle_{n=1}^\infty)$ be J -set. Then by Theorem 1.10 we have $J(\mathbb{N}) \cap \text{clFS}(\langle x_n \rangle_{n=1}^\infty) \neq \emptyset$. We choose $p \in J(\mathbb{N}) \cap \text{clFS}(\langle x_n \rangle_{n=1}^\infty)$. By

[11, Lemma 5.11], $\bigcap_{m=1}^{\infty} \text{clFS}(\langle x_n \rangle_{n=m}^{\infty})$ is a subsemigroup of $\beta\mathbb{N}$. First we show that for each $m \in \mathbb{N}$, $J(\mathbb{N}) \cap \text{clFS}(\langle x_n \rangle_{n=m}^{\infty}) \neq \emptyset$. For this, let $m \in \mathbb{N}$ with $m > 1$. Then $FS(\langle x_n \rangle_{n=1}^{\infty}) = FS(\langle x_n \rangle_{n=m}^{\infty}) \cup FS(\langle x_n \rangle_{n=1}^{m-1}) \cup \bigcup \{t + FS(\langle x_n \rangle_{n=m}^{\infty}) : t \in FS(\langle x_n \rangle_{n=1}^{m-1})\}$. So we must have one of

- (1) $FS(\langle x_n \rangle_{n=1}^{m-1}) \in p$,
- (2) $FS(\langle x_n \rangle_{n=m}^{\infty}) \in p$,
- (3) $t + FS(\langle x_n \rangle_{n=m}^{\infty}) \in p$ for some $t \in FS(\langle x_n \rangle_{n=1}^{m-1})$.

Clearly (1) does not hold, because in that case p becomes a member of \mathbb{N} that is a principal ultrafilter, while it is a member $p \in \beta\mathbb{N} \setminus \mathbb{N}$. If (2) holds then we have done. So let (3) holds, that is there exists $t + FS(\langle x_n \rangle_{n=m}^{\infty}) \in p$ for some $t \in FS(\langle x_n \rangle_{n=1}^{m-1})$. We choose some $q \in \text{clFS}(\langle x_n \rangle_{n=m}^{\infty})$ so that $t + q = p$. For every $F \in q$ we have $t \in \{n \in \mathbb{N} : -n + (t + F) \in q\}$ so that $t + F \in p$. Since J -sets in $(\mathbb{N}, +)$ are translation invariant F becomes a J -set. This implies that $q \in J(S)$, so that $q \in J(\mathbb{N}) \cap \text{clFS}(\langle x_n \rangle_{n=m}^{\infty})$ for all $m \in \mathbb{N}$.

(a) \Rightarrow (c) Since $\langle x_n \rangle_{n=1}^{\infty}$ is an almost minimal sequence we have $J(\mathbb{N}) \cap \bigcap_{n=m}^{\infty} \text{clFS}(\langle x_n \rangle_{n=m}^{\infty}) \neq \emptyset$. Again $J(\mathbb{N})$ being closed ideal, we can conclude that and so $J(\mathbb{N}) \cap \bigcap_{n=m}^{\infty} \text{clFS}(\langle x_n \rangle_{n=m}^{\infty})$ is a compact right topological semigroup so that it contains idempotents.

(c) \Rightarrow (a) is obvious. □

Lemma 2.8. *If A is a C -set in $(\mathbb{N}, +)$ then both nA and $n^{-1}A$ are also C -sets for any $n \in \mathbb{N}$, where $n^{-1}A = \{m \in \mathbb{N} : n \cdot m \in A\}$.*

Proof. [14, Lemma 8.1] □

Lemma 2.9. *If A is a C^* -set in $(\mathbb{N}, +)$ then $n^{-1}A$ is also C^* -set for any $n \in \mathbb{N}$.*

Proof. Let A be a C^* -set and $t \in \mathbb{N}$. To prove that $t^{-1}A$ is a C^* -set it is sufficient to show that for any C -set C , $C \cap t^{-1}A \neq \emptyset$. Since C is C -set tC is also C -set so that $A \cap tC \neq \emptyset$. Choose $n \in tC \cap A$ and $k \in C$ such that $n = tk$. Therefore $k = n/t \in t^{-1}A$. Hence $C \cap t^{-1}A \neq \emptyset$. □

Theorem 2.10. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be an almost minimal sequence and A be a C^* -set $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that*

$$FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A.$$

Proof. Since $\langle x_n \rangle_{n=1}^{\infty}$ is an almost minimal sequence by Theorem 2.7 it follows that we can find some idempotent $p \in J(\mathbb{N})$ for which $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$. Again, since A is a C^* -set $(\mathbb{N}, +)$, by the above Lemma 2.9 for every $s \in \mathbb{N}$, $s^{-1}A \in p$. Let $A^* = \{s \in A : -s + A \in p\}$. Then by [11, Lemma 4.14] $A^* \in p$. We can choose $y_1 \in A^* \cap FS(\langle x_n \rangle_{n=1}^{\infty})$. Inductively let $m \in \mathbb{N}$ and $\langle y_i \rangle_{i=1}^m, \langle H_i \rangle_{i=1}^m$ in $\mathcal{P}_f(\mathbb{N})$ be chosen with the following properties:

- (1) $i \in \{1, 2, \dots, m-1\}$ $\max H_i < \min H_{i+1}$;

(2) If $y_i = \sum_{t \in H_i} x_t$ then $\sum_{t \in H_m} x_t \in A^*$ and $FP(\langle y_i \rangle_{i=1}^m) \subseteq A$.

We observe that $\{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\} \in p$. Let $B = \{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\}$, let $E_1 = FS(\langle y_i \rangle_{i=1}^m)$ and $E_2 = FP(\langle y_i \rangle_{i=1}^m)$. Now consider

$$D = B \cap A^* \cap \bigcap_{s \in E_1} (-s + A^*) \cap \bigcap_{s \in E_2} (s^{-1} A^*)$$

Then $D \in p$. Now choose $y_{m+1} \in D$ and $H_{m+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{m+1} > \max H_m$. Putting $y_{m+1} = \sum_{t \in H_{m+1}} x_t$ shows that the induction can be continued and proves the theorem. \square

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